The Optimal Shape of a Javelin

A dynamical systems approach using a similarity solution

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Similarity, Vancouver

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Overview

- Start with an optimization problem
- Derive nonlinear, differential, eigenvalue equation
- Difficulty is due to singular boundary condition
- Using a similarity solution we “peel away” singularity
- Locate stable manifold to decrease shooting dimension
- Shoot for boundary condition with no numerical difficulties

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Maximize the lowest resonant frequency of unsupported rod

- Minimize harmonic resonance
- Minimize vibration amplitude for given energy
- Maximize damping rate
- Understand design limitations

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Subject to the following constraints

- Fixed cross-sectional *shape*
- Fixed length, L
- Fixed volume, V
- Uniform material, with Young's modulus E, density ρ

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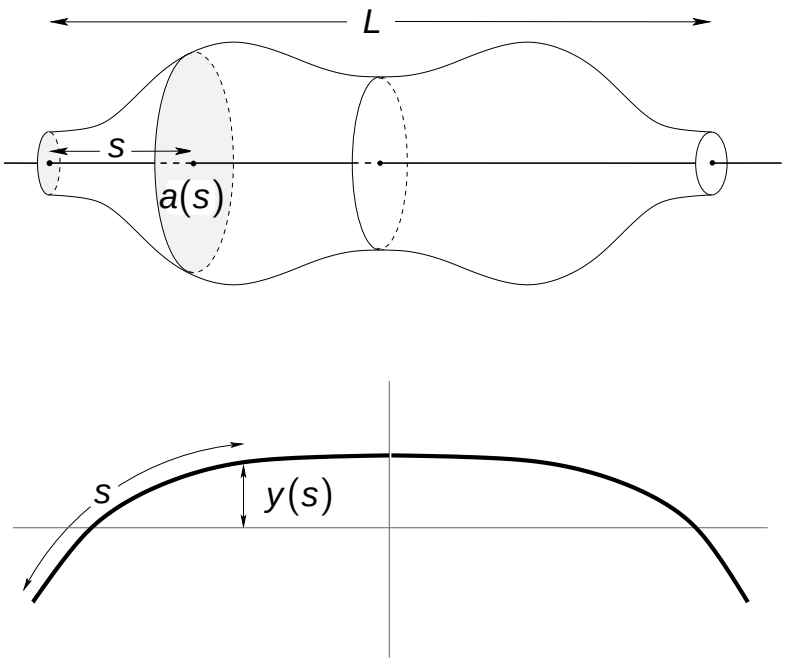
The setup

- Distance from the tip, s
- The cross-sectional area, $a(s)$
- The deflection of the neutral axis, $y(s)$
- Assume thin rod, $\sqrt{a} \ll L$;
- and small deflection, $y \ll L$



Scaled average Lagrangian

area	length	energy
$[a] = \frac{V}{L}$	$[s] = \frac{L}{2}$	$[\overline{\mathfrak{L}}] = \frac{8cEV^2}{L^5}$

$$\overline{\mathfrak{L}}[y] = \int_0^2 \lambda^2 a y^2 - a y_{ss}^2 \, ds, \qquad \int_0^2 a \, ds = 2$$

After Euler-Lagrange, and $\lambda-$maximization

Definition of the torque, φ	$\varphi - a^2 y_{ss} = 0$
Vibration problem	$\varphi_{ss} - \lambda^2 a y = 0$
$\lambda-$maximization	$2\left(\frac{\varphi^2}{a^3}\right)_s - \lambda^2 \left(y^2\right)_s = 0$
Volume constraint	$b_s - a = 0$

$$\left.\begin{aligned} \varphi &= 0 \\ \varphi_s &= 0 \\ b &= 0 \end{aligned}\right\} \begin{array}{l} \text{at tip} \\ s = 0 \end{array} \qquad \left.\begin{aligned} y_s &= 0 \\ a_s &= 0 \\ b &= 1 \end{aligned}\right\} \begin{array}{l} \text{at center} \\ s = 1 \end{array}$$

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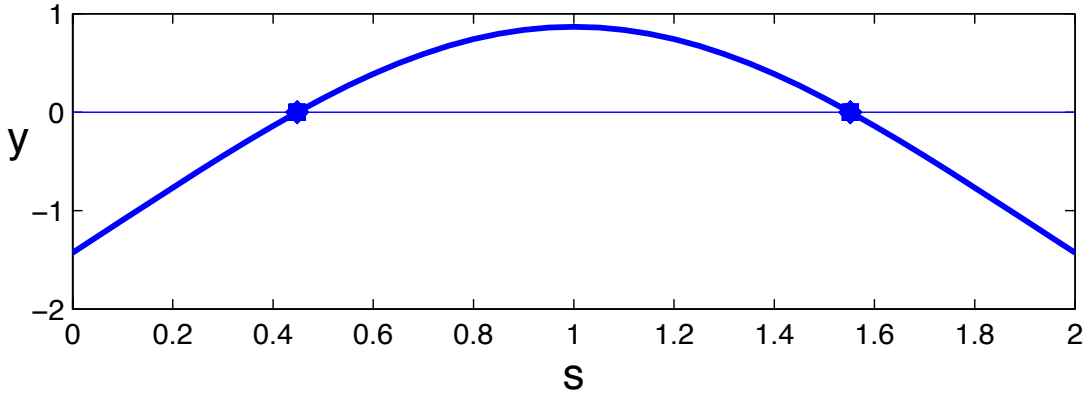
Baseline example

For a constant cross-section, $a \equiv 1$, the vibration eigenvalue problem has a simple analytic solution



$\lambda \approx 5.5$

$$y(s) = \frac{\cos(\sqrt{\lambda}(s-1))}{\cos(\sqrt{\lambda})} + \frac{\cosh(\sqrt{\lambda}(s-1))}{\cosh(\sqrt{\lambda})}$$

Difficulties of the full problem

- It is singular at the free end, $s = 0$
- We expect the tip of the rod to taper to a point
- Straightforward shooting is not possible
- Backward shooting is also not possible

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The plan:

- Find algebraic solution from balance of units
- Divide each variable by its respective algebraic solution
- Find an ODE for the ratios
- Hope that this ODE is nicer
- Solve starting near a critical point

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Let A, B, Y, P, and S be the "units" of a, b, y, φ and s.
The 4 equations of the ODE system give

Unit Balance

$$\varphi - a^2 y_{ss} = 0 \quad \Longrightarrow \quad P = A^2 Y S^{-2}$$

$$\varphi_{ss} - \lambda^2 a y = 0 \quad \Longrightarrow \quad P S^{-2} = A Y$$

$$2\left(\frac{\varphi^2}{a^3}\right)_s - \lambda^2 \left(y^2\right)_s = 0 \quad \Longrightarrow \quad P^2 A^{-3} S^{-1} = Y^2 S^{-1}$$

$$b_s - a = 0 \quad \Longrightarrow \quad B S^{-1} = A$$

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Solving these equations gives the relations

$$A = S^4$$

$$B = S^5$$

$$P = Y S^6$$

The relations hint to a solution of the form

$$\tilde{a}(s) = a_0 s^4$$
$$\tilde{b}(s) = b_0 s^5$$
$$\tilde{\varphi}(s) = \varphi_0 s^{p+6}$$
$$\tilde{y}(s) = y_0 s^p$$

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The only (physically viable) algebraic solution is

$$\hat{a}(s) = \frac{\lambda^2}{72} s^4 \qquad \hat{b}(s) = \frac{\lambda^2}{360} s^5$$
$$\hat{\varphi}(s) = y_0 \frac{\lambda^4}{864} s^4 \qquad \hat{y}(s) = y_0 s^{-2}$$

which also satisfies the boundary conditions at the tip, $s = 0$, but not at the center, $s = 1$

New variables: $\alpha, \beta, \psi, \zeta$

$$\alpha = \frac{a}{\hat{a}} \qquad \beta = \frac{b}{\hat{b}}$$

$$\psi = \frac{\varphi}{\hat{\varphi}} \qquad \zeta = \frac{y}{\hat{y}}$$

while the independent variable is $t = -\log s$

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The ODE for the quotients is

$$6\psi - \alpha^2(\partial+2)(\partial+3)\zeta = 0$$

$$(\partial-4)(\partial-3)\psi - 12\alpha\zeta = 0$$

$$(4+\partial)\frac{\psi^2}{\alpha^3} - 2\zeta(2+\partial)\zeta = 0$$

$$(5-\partial)\beta - 5\alpha = 0$$

where $\partial = \frac{d}{dt}$

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The ODE for the quotients is

No λ!

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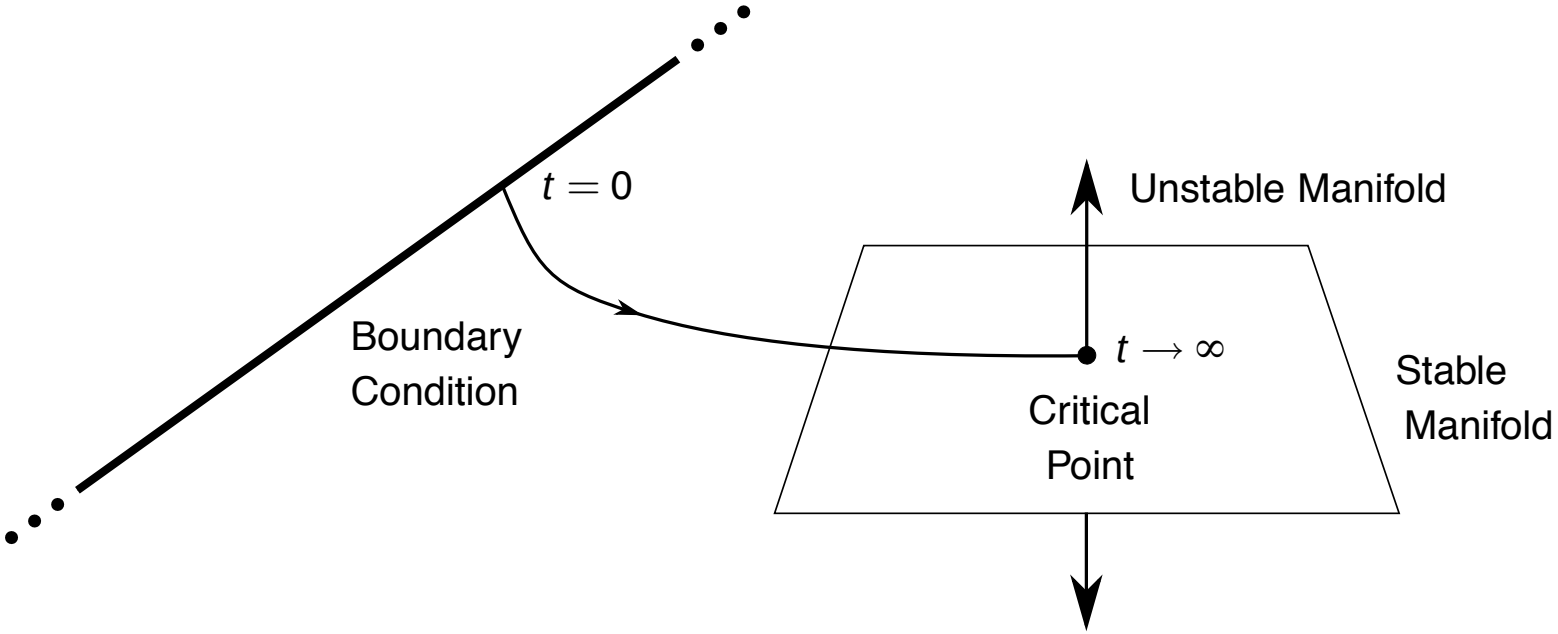
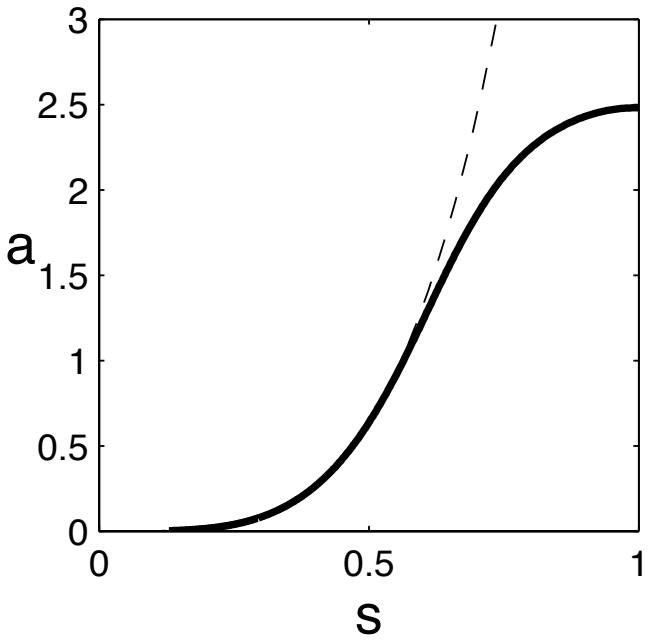
New system

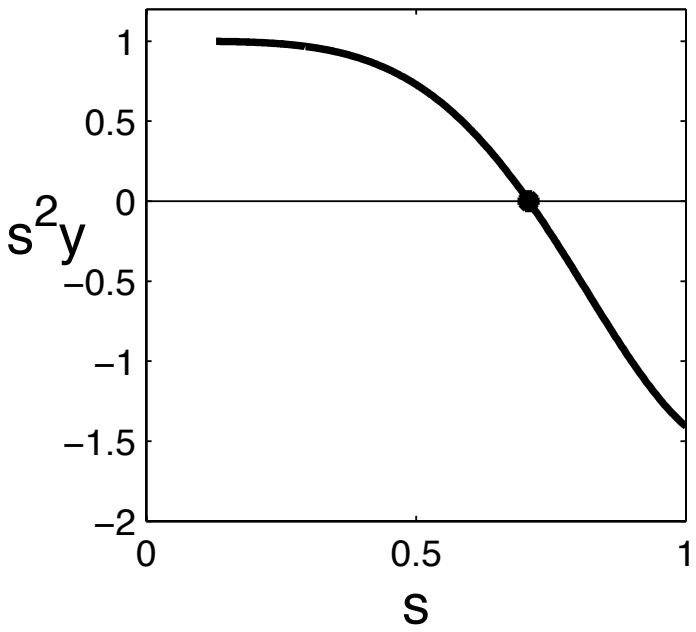
- The eigenvalue λ, is not present in the new equations
- It appears only in a boundary condition
- Other boundary condition is satisfied at the critical point
- Reduces to an easy shooting problem



Resulting taper

- Numerical solution has $\lambda \approx 27$, five times larger than the untapered rod
- No numerical difficulties are encountered
- Easy to solve for different setup: cantilever, hinge, etc





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Conclusions

- Unstable, 4D shooting is replaced by regular 2D shooting
- Eigenvalue is removed from evolution equations
- Similarity solution gives analytic information near singularity
- Method seems to be general (solves “tallest column” as well)

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Thank-you!!

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