

Midterm 1 Solution

1. a. If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest values possible for $\|\vec{v} - \vec{w}\|$? What are the largest and smallest values of $\vec{v} \cdot \vec{w}$?

Solution: Using the triangle inequality we have

$$\|\vec{v} - \vec{w}\| \leq \|\vec{v}\| + \|-\vec{w}\| = 3 + 5 = 8.$$

On the other hand, since $\vec{v} = \vec{w} + (\vec{v} - \vec{w})$, we can write the triangle inequality for the plus sign in this equation:

$$\begin{aligned} \|\vec{v}\| = \|\vec{w} + (\vec{v} - \vec{w})\| &\leq \|\vec{w}\| + \|\vec{v} - \vec{w}\| \\ 5 &\leq 3 + \|\vec{v} - \vec{w}\| \Rightarrow \|\vec{v} - \vec{w}\| \geq 2. \end{aligned}$$

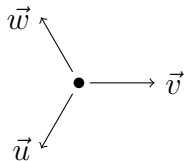
These inequalities are “tight” as can be seen from taking \vec{v} and \vec{w} in the same direction (for minimum) and in opposite directions (for maximum).

Using the cosine formula we have

$$\begin{aligned} \vec{v} \cdot \vec{w} &= \|\vec{v}\| \|\vec{w}\| \cos \alpha \\ &= 15 \cos \alpha \Rightarrow -15 \leq \vec{v} \cdot \vec{w} \leq 15. \end{aligned}$$

- b. Can three vectors in the xy plane \mathbb{R}^2 , have $\vec{u} \cdot \vec{v} < 0$, $\vec{v} \cdot \vec{w} < 0$, $\vec{w} \cdot \vec{u} < 0$?

Solution: Yes. For this we can take, for example, three vectors making 120° with each other:



If the lengths are all 1, the dot products are all $\cos 120^\circ = -\frac{1}{2}$.

- c. How long is the vector $\vec{v} = (1, 1, 1, \dots, 1) \in \mathbb{R}^9$? Find a unit vector \vec{u} in the same direction as \vec{v} and a unit vector \vec{w} that is perpendicular to \vec{v} .

Solution: Since $\|\vec{v}\|^2 = \underbrace{1^2 + 1^2 + \dots + 1^2 + 1^2}_{9 \text{ times}} = 9$, we have that $\|\vec{v}\| = 3$. So, since the norm

is linear, we can define $\vec{u} = \vec{v}/\|\vec{v}\| = \vec{v}/3 = (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$. Since $(1, -1, 0, 0, \dots, 0) \perp \vec{v}$, we only need to make it a unit vector, so we define $\vec{w} = (1, -1, 0, 0, \dots, 0)/\|(1, -1, 0, 0, \dots, 0)\| = (\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}}, 0, 0, \dots, 0)$.

2. a. (10 pts) Find a combination $x_1\vec{w}_1 + x_2\vec{w}_2 + x_3\vec{w}_3$ that gives the zero vector:

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \vec{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Solution: Since the solution does not change when we eliminate, we go ahead with elimination:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 5 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the third component is zero in the three eliminated column, the third component of any combination of them will always be zero. Thus, we have 3 unknowns and only 2 equations. We can choose $x_3 = 1$ and look for a solution for the remaining 2 variables. The second row reads $-3x_2 - 6x_3 = 0$. Since we assumed $x_3 = 1$, we get that $x_2 = -2$. Similarly, we get from the first equation that $x_1 = 1$. So a solution is

$$x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

b. *Solution:*

- Those vectors are linearly dependent. (We could not find a third non-zero pivot.)
- They lie in a plane. (We could eliminate the third coordinate.)
- The matrix W with these column is not invertible. (We could find a non-zero solution to the problem.)

c. (6 pts) Must the rows of the matrix W also combine to give zero? If so, find one such linear combination.

Solution: The elimination process changes the last row into a row of zeros, therefore, it is clear that indeed, the rows have a non-zero linear combination. We can find it systematically by following what we did in the elimination process. However, we need to be careful since we change the second row before adding it to the third: $R_3 - 3R_1 - (2(R_2 - 2R_1)) = 0$, therefore, $R_3 - 2R_2 + R_1$.

3. Let A be the matrix in the following $A\vec{x} = \vec{b}$ system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$

a. (10 pts) For which values of k will the system have a unique solution? (Use augmented matrix).

Solution: Using the augmented matrix notation we eliminate:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 3 \\ 3 & 4 & k & 7 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & k-3 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-5 & 0 \end{array} \right]$$

The last line of the last system means that $x_3(k - 5) = 0$, therefore, if $k \neq 5$ there is only one solution for x_3 (zero). Since the other variables have a found a pivot, there is only one solution.

b. (5 pts) For which values of k will there be many solutions?

Solution: Using the same eliminated matrix see that if $k = 5$ the last row is satisfied regardless of the value of x . So for $k = 5$ there are many solutions.

c. (10 pts) For $k = 4$, find the LU decomposition of A , $A = LU$.

Solution: We needed the multipliers 1, 3 in the first elimination and 1 in the second. The matrix U we get directly from the eliminated matrix by setting $k = 4$: Therefore the matrices are:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

4. a. (5 pts) What is the 2 by 2 identity matrix I ? $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. (5 pts) What is the 2 by 2 exchange matrix P ? $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c. (5 pts) What 2 by 2 matrix R_{90} , rotates every vector by 90° ? $R_{90} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

d. (5 pts) What 2 by 2 matrix R_{180} , rotates every vector by 180° ? $R_{180} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

e. (5 pts) What 2 by 2 matrix $E_{2 \times 2}$ subtracts the first component from the second one? $E_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

f. (Bonus) What 3 by 3 matrix $E_{3 \times 3}$ does the same? $E_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}, \quad R_{90} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}, \quad R_{180} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}, \quad E \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y - x \end{bmatrix}.$$